SEMINAR 1: **PORTFOLIO THEORY**

FINANCIAL ECONOMICS

April 27, 2021

1. Exam December 2009/10

Let us assume a one-period economy with 2 risky assets, A and B. Asset A has a rate of return with mean 20% and standard deviation (or volatility) 35%. Asset B has a rate of return with mean 10% and volatility 15%. The correlation between both returns is 0.5.

(a) If you had €1,000 and you wanted a 15% expected return, how would you invest your money in both assets? Which is the return volatility of that portfolio?

(b) If you had €1,000 and you wanted your portfolio return to have a volatility of 20%, in which 2 portfolios of both assets could you invest your money? Which of the 2 portfolios would an investor with mean-variance preferences choose?

Hint : The solution of $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

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1. Exam December 2009/10

Construct a portfolio by investing a fraction ω_A of your wealth in asset A and a fraction $\omega_B = 1 - \omega_A$ of your wealth in asset B such that

$$\mathbb{E}[R_P] = \omega_A \mathbb{E}[R_A] + (1 - \omega_A)\mathbb{E}[R_B] = 15\%$$

$$\rightarrow \omega_A = \frac{15\% - \mathbb{E}[R_B]}{\mathbb{E}[R_A] - \mathbb{E}[R_B]} = \frac{15\% - 10\%}{20\% - 10\%} = 0,5$$

$\implies$ **Invest €500 in asset A and €500 in asset B**

The return volatility of this portfolio is given by

$$\sigma_P = \sqrt{(\omega_A \sigma_A)^2 + (\omega_B \sigma_B)^2 + 2\omega_A \omega_B \underbrace{cov(R_A, R_B)}_{=\sigma_A \sigma_B \rho_{(A,B)}}}$$

$$\sigma_P = \sqrt{306,25 + 56,25 + 131,25} = \sqrt{493,75} = 22,22\%$$

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1. Exam December 2009/10 $\rightarrow$ (b)

Construct a portfolio by investing a fraction ω_A of your wealth in asset A and a fraction $\omega_B = 1 - \omega_A$ of your wealth in asset B such that $\sigma_P = 20\%$. Then,

$$\sigma_P^2 = (\omega_A \sigma_A)^2 + (\omega_B \sigma_B)^2 + 2\omega_A \omega_B \sigma_A \sigma_B \rho_{(A,B)} = 400(\%^2)$$

Rearranging terms and taking into account that $\omega_B = 1 - \omega_A$

$$\sigma_P^2 = \left(\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{(A,B)}\right)\omega_A^2 + 2\left(\sigma_A\sigma_B\rho_{(A,B)} - \sigma_B^2\right)\omega_A + \sigma_B^2 = 400(\%^2)$$

If we substitute the values provided

$$925\omega_A^2 + 75\omega_A - 175 = 0 \implies \omega_A = \frac{-75 \pm \sqrt{75^2 + 4(925)(175)}}{2(925)}$$

The solutions to this equation is given by

$$\begin{cases} \text{Portfolio 1} \implies \omega_A^1 = 0,396 \\ \text{Portfolio 2} \implies \omega_A^2 = -0,477 \end{cases}$$

1. Exam December 2009/10 $\rightarrow$ (b)

The expected return and volatility of portfolio 1 and 2

$$\textbf{Portfolio 1}\ (\omega_A^1 = 0,396) \quad \rightarrow \quad \mathbb{E}\left[R_{P_1}\right] = 14\% \quad \text{and} \quad \sigma_{P_1} = 20\%$$

$$\textbf{Portfolio 2}\ (\omega_A^2 = -0,477) \quad \rightarrow \quad \mathbb{E}\left[R_{P_2}\right] = 5\% \quad \text{and} \quad \sigma_{P_2} = 20\%$$

$\Longrightarrow$ Portfolio 1 has same volatility but higher expected return than Portfolio 2

$\Longrightarrow$ The mean-variance investor invests in

$$\textbf{Portfolio 1} \rightarrow \begin{cases} €396 \text{ in asset A} \\ €604 \text{ in asset B} \end{cases}$$

2. September Exam 2009/10

Consider an economy with one safe asset and one risky asset. The safe asset is a zero coupon bond maturing in 1 year with a nominal of €100 and a price of €95,24. The risky asset is a stock with a price today of €90,91, and whose value in $t = 1$ is a random variable with mean €100 and standard deviation €9,09.

(a) Translate the previous information into returns. In other words, show that the annual interest rate of the bond is 5%, and the stock return has an expected value of 10% and a standard deviation (or volatility) of 10%.

(b) Draw the efficient mean-variance frontier of this economy.

(c) What is the Sharpe ratio of the risky efficient portfolios?

(d) A portfolio invests 150% in the bond and -50% in stock. Is this efficient?

(e) Now consider instead an economy without the riskless asset , but with many risky assets (for example, shares in other companies). Assume that the correlation between their returns is 0. What happens to the return volatility of a portfolio with equal weight in every risky asset as we add more and more assets to it?

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2. September Exam 2009/10 $\rightarrow$ (a)

- **Safe asset**

  $\rightarrow$ Expected return: $\mathbb{E}[R_s] = R_s = \frac{€100 - €95,24}{€95,24} = 5\%$

  $\rightarrow$ Volatility: $\sigma_s = 0$

- **Risky asset**

  $\rightarrow$ Expected return: $\mathbb{E}[R_r] = \frac{€100 - €90,91}{€90,91} = 10\%$

  $\rightarrow$ Volatility: $\sigma_r = \frac{€9,09}{€90,91} = 10\%$

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2. September Exam 2009/10 $\rightarrow$ (b)

The expected return and volatility of a portfolio constructed by investing a fraction ω_r in the risky asset and a fraction ω_s in the safe asset

$$\mathbb{E}\left[R_p\right] = (1-\omega_r)R_s + \omega_r \mathbb{E}\left[R_r\right] = R_s + \omega_r\left[\mathbb{E}\left[R_r\right] - R_s\right] \quad (1a)$$

$$\sigma_P = \omega_r \sigma_r \rightarrow \text{ given that } \sigma_s = 0 \quad (1b)$$

From (1b) we know that $\omega_r = \frac{\sigma_P}{\sigma_r}$, and substituting this expression in (1a)

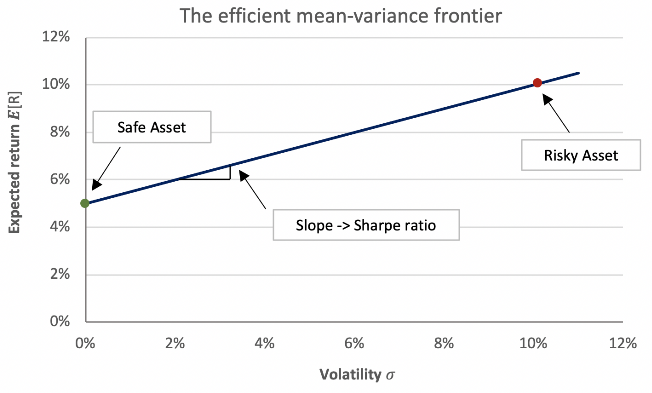
The efficient mean-variance frontier $\rightarrow \mathbb{E}\left[R_p\right] = R_s + \underbrace{\left[\frac{\mathbb{E}\left[R_r\right] - R_s}{\sigma_r}\right]}_{\text{Sharpe ratio}} \sigma_P$

2. September Exam 2009/10 → (c)

Plugging the values provided

$$\mathbb{E}\left[R_p\right] = 5\% + 0,5\sigma_P$$

→ Sharpe ratio of the risky efficient portfolios = 0,5



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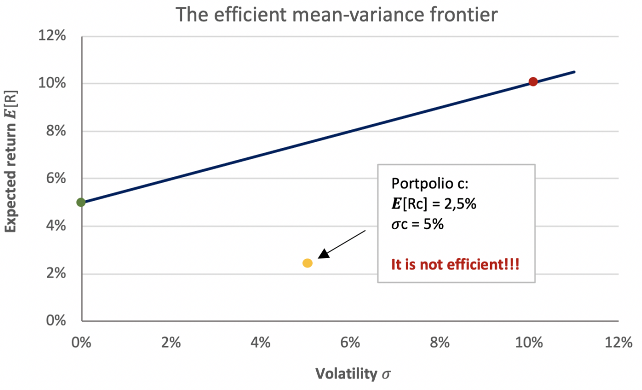
2. September Exam 2009/10 → (d)

A portfolio C that invests $\omega_s = 1,5$ and $\omega_r = -0,5$

$$\rightarrow \mathbb{E}\left[R_{P_c}\right] = 2,5\% \text{ and } \sigma_{P_c} = 5\%$$



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2. September Exam 2009/10 $\rightarrow$ (e)

As we add more and more assets to a portfolio with equal weight in every risky asset (uncorrelated assets), the return **volatility does not only decrease but it tends to zero**.

Proof: the variance of a portfolio constructed with $j = 1, ..., J$ assets

$$\sigma_P^2 = \sum_{j=1}^{J} \left(\frac{1}{J}\right)^2 \sigma_j^2 + \sum_{i=1}^{J} \sum_{j=1, j \neq i}^{J} \left(\frac{1}{J}\right)^2 \sigma_{ij}$$

Given that correlation between assets is zero $\rightarrow \sigma_{ij} = 0$

$$\sigma_P^2 = \frac{1}{J} \left[\frac{1}{J} \sum_{j=1}^{J} \sigma_j^2\right] = \frac{1}{J} \text{ (average variance of individual stocks)}$$

As $J \rightarrow \infty$ volatily tends to zero $\sigma_P \rightarrow 0$

3. September Exam 2009/10

Consider an economy with two dates (1 period), with two risky assets, A and B. Asset A has a rate of return with mean 12% and a volatility of 20%, while asset B has a rate of return with mean 15% and a volatility of 25%. The correlation between the returns of asset A and asset B is 1.

(a) Express the expected return on a portfolio composed of the two risky assets as a function of the expected returns on each asset. Do the same for the volatility (standard deviation) of the portfolio (as function of the volatilities of each asset). Represent the efficient frontier in the mean-standard deviation space. Locate assets A and B on the graph.

(b) Assume the correlation between the returns on asset A and the returns on asset B is 0 (not 1 as before!). Compute the expected return, volatility and the Sharpe ratio of a portfolio, M, that invests 52.23% in asset A and the rest in asset B if we introduce a bond paying a risk-free interest rate of 5%.

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3. September Exam 2009/10$\rightarrow$ (a)

The expected return and volatility of a portfolio constructed by investing a fraction ω_A in assets A and a fraction ω_B in asset B

$$\mathbb{E}[R_P] = \omega_A \mathbb{E}[R_A] + (1-\omega_A)\mathbb{E}[R_B]$$
$$\sigma_P^2 = (\omega_A\sigma_A)^2 + (\omega_B\sigma_B)^2 + 2\omega_A\omega_B\sigma_A\sigma_B\rho_{(A,B)}$$
$$\text{given that } \rho_{(A,B)} = 1 \rightarrow \sigma_P = \omega_A\sigma_A + (1-\omega_A)\sigma_B$$

Then, $\omega_A = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}$ and substituting this expression in the first equation $\rightarrow$

The efficient mean-variance frontier

$$\mathbb{E}[R_p] = \underbrace{\mathbb{E}[R_B] - \frac{\mathbb{E}[R_A] - \mathbb{E}[R_B]}{\sigma_A - \sigma_B}\sigma_B}_{=0} + \underbrace{\frac{\mathbb{E}[R_A] - \mathbb{E}[R_B]}{\sigma_A - \sigma_B}}_{=0,6}\sigma_P$$

3. September Exam 2009/10 → (a)

Plugging the values provided → $\mathbb{E}\left[R_p\right] = 0,6\sigma_P$

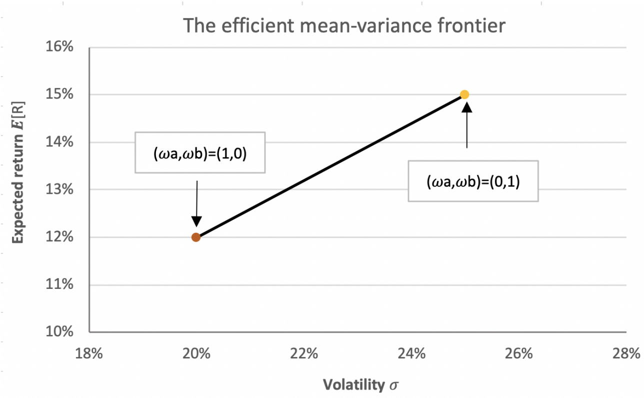

Figure 1: With short-selling constraint

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3. September Exam 2009/10→ (b)

The expected return and volatility of a portfolio M constructed by investing a fraction $\omega_A = 52,23\%$ and $\omega_B = 47,77\%$

$$\mathbb{E}[R_M] = \omega_A \mathbb{E}[R_A] + (1-\omega_A)\mathbb{E}[R_B] = 0,52x12 + 0,48x15 = 13,43\%$$

$$\sigma_M = \sqrt{(\omega_A\sigma_A)^2 + (\omega_B\sigma_B)^2} = \sqrt{(0,52x20)^2 + (0,48x25)^2} = 15,87\%$$

If the return of the safe asset $R_s = 5\%$, then the Sharpe ratio of portfolio M

$$\frac{\mathbb{E}[R_M] - R_s}{\sigma_M} = \frac{13,43\% - 5\%}{15,83\%} = 0,53$$

4. July 2016

Consider an economy with one period and two dates ($t = 0$, $t = 1$), a bond (asset 0) and two risky assets (assets 1 and 2). The risk-free rate paid by the bond (interest rate) is $r = 5\%$. The risky asset returns and characteristics are given in the table below:

	$\mathbb{E}[R_i]$	σ_i
Asset 1	6,08	6
Asset 2	15,88	23,32

Correlation between risky asset returns $= \rho_{12} = 0$

(a) Which of the following portfolios is the tangency portfolio?

(i) Portfolio A: 20% in Asset 1, 80% in Asset 2
(ii) Portfolio B: 60% in Asset 1, 40% in Asset 2
(iii) Portfolio C: 80% in Asset 1, 20% in Asset 2

(b) Calculate the efficient portfolio for an expected return of 12%. Define clearly the investment position in each of the 3 assets.

(c) What is the expected return of an efficient portfolio with the same volatility as asset 1 ($\varepsilon = 6\%$)?

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4. July 2016 → (a)

The expected return, volatility and Sharpe ratio of each of theses portfolios

- **Portfolio A** $\rightarrow (\omega_1, \omega_2) = (0,2; 0,8)$

$$\mathbb{E}[R_A] = 0,2x6,08 + 0,8x15,88 = 13,9\%$$

$$\sigma_A = \sqrt{(0,2x6)^2 + (0,8x23,32)^2} = 18,7\%$$

$$\text{Sharp ratio}_A = \frac{13,9 - 5}{18,7} = 0,48$$

- **Portfolio B** $\rightarrow (\omega_1, \omega_2) = (0,6; 0,4)$

$$\mathbb{E}[R_B] = 0,6x6,08 + 0,4x15,88 = 10\%$$

$$\sigma_B = \sqrt{(0,6x6)^2 + (0,4x23,32)^2} = 10\%$$

$$\text{Sharp ratio}_B = \frac{10 - 5}{10} = 0,5$$

- **Portfolio C** $\rightarrow (\omega_1, \omega_2) = (0,8; 0,2)$

$$\mathbb{E}[R_C] = (0,8)6,08 + (0,2)15,88 = 8,04\%$$

$$\sigma_C = \sqrt{((0,8)6)^2 + ((0,2)23,32)^2} = 6,7\%$$

$$\text{Sharp ratio}_C = \frac{8,04 - 5}{6,7} = 0,45$$

4. July 2016 $\rightarrow$ (a)

The tangency portfolio is given by the portfolio with the highest Sharpe ratio:

$$\textbf{Tangency portfolio} \rightarrow \begin{cases} (\omega_1, \omega_2) = (0,6; 0,4) \\ \mathbb{E}\left[R_T\right] = 10\% \text{ and } \sigma_T = 10\% \end{cases}$$

The weights (ω_0, ω_T) of an efficient portfolio with expected return of 12%

$$\mathbb{E}\left[R_E\right] = \omega_0 R_0 + (1 - \omega_0)\mathbb{E}\left[R_T\right] = 12\% \implies \omega_0 = \frac{12\% - 10\%}{5\% - 10\%} = -0,4$$

$$\textbf{The weights of the efficient portfolio} \implies \begin{cases} \omega_0 = -0,4 \\ \omega_1 = 1,4x0,6 = 0,84 \\ \omega_2 = 1,4x0,4 = 0,56 \end{cases}$$

4. July 2016→ (b)

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4. July 2016$\rightarrow$ (c)

Any efficient portfolio has to be on the efficient frontier

$$\mathbb{E}\left[R_P\right] = R_0 + \underbrace{\frac{\mathbb{E}\left[R_T - R_0\right]}{\sigma_T}}_{\text{Sharpe ratio}} \sigma_P = 5\% + 0,5\sigma_P$$

Then, the expected return of an efficient portfolio with volatility of 6%

$$\mathbb{E}\left[R_P\right] = 5\% + 0,5x6\% = 8\%$$