SEMINAR 6: **DERIVATIVES, PART C**

FINANCIAL ECONOMICS

June 2020

1. Exam June 2011

Consider an economy with two dates t = 0 & 1, with two assets: asset X and asset Y. At t = 1 there are 2 states of nature: state A and state B. The evolution of the asset prices in time and conditional on the states of nature is the following (in €):

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0

There is an organized market for both assets, thus they can be bought or sold (including short-selling) at any moment at these prices.

(a) If a new zero-coupon risk-free bond is created, with a nominal value of €150 and maturity at time t = 1, which would be its no-arbitrage price at t = 0?

1. Part a) solution

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
Bond	P_B=?	150	150

1. Find a replicating portfolio for the bond

$$\text{State A:} \quad 150zX + 50zY = 150 \tag{1}$$
$$\text{State B:} \quad 90zX = 150 \tag{2}$$

$$\text{From (2)}: \quad zX = 1.6667$$
$$\text{Plugging into (1)}: \quad zY = \frac{150 - 150 * 1.6667}{50} = -2$$

Replicating portfolio Z: long 1.6667 units of X and short 2 units of Y

2. Bond price equals the price of this replicating portfolio

$$P_B = 100 * 1.6667 + 10 * (-2) = 146.67$$

1. Part b)

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0

(b) Suppose that a call option (of the European type) starts being traded, having as an underlying asset the asset X, with a maturity at time t = 1 and exercise price of €80. What will be the no-arbitrage price of the call at t =0?

1. Part b) solution

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
C_X	P_C=?	70	10

- Let's describe the payoffs from this call option with strike price $K = 80$ in each state
- In both states $S_T > K$ so you would use the call (buy cheap at call price, sell expensively in the market)
  - State A: buy X for 80 (strike price), sell for 150. Payoff = 150 - 80 = 70
  - State B: buy X for 80 (strike price), sell for 90. Payoff=90 - 80 = 10
- As with part a), let's construct a replicating portfolio with these payoffs and then price the option using this portfolio

1. Part b) solution

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
C_X	P_C=?	70	10

- Replicating portfolio:

$$\text{State A:} \quad 150zX + 50zY = 70 \tag{3}$$

$$\text{State B:} \quad 90zX = 10 \tag{4}$$

$$\text{From (4)}: \quad zX = 0.1111$$

$$\text{Plugging into (3)}: \quad zY = \frac{70 - 150 * 0.1111}{50} = 1.0667$$

Replicating portfolio Z: long 0.1111 units of X and 1.0667 units of Y

- Option price equals the price of this replicating portfolio

$$P_C = 100 * 0.1111 + 10 * 1.0667 = 21.778$$

1. Part c)

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
C_X	22	70	10

c) Assume that the call from part b) is traded in the market at a price of € 22. Discuss if there is an arbitrage opportunity or not. If there is, specify into detail an arbitrage strategy (specify which assets – and in what quantities – you would buy and sell, counting with the assets X and Y) as well as the payoffs or cash-flows of this strategy in each date and state of nature

- We know the call is too expensive relative to its no-arbitrage price of 21.778, which we derived in part b)
- Arbitrage strategy: short call, long replicating portfolio

1. Part c) solution

c) Assume that the call from part b) is traded in the market at a price of € 22. Discuss if there is an arbitrage opportunity or not. If there is, specify into detail an arbitrage strategy (specify which assets – and in what quantities – you would buy and sell, counting with the assets X and Y) as well as the payoffs or cash-flows of this strategy in each date and state of nature

- Arbitrage strategy

	t=0	t=1; state A	t=1; state B
Long RP: 0.11X + 1.67Y	-21.778	70	10
Short C_X	22	-70	-10
Payoff	0.22	0	0

1. Part d)

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
Call, K=80 (C_X)	21.778	70	10
Put, K=80 (P_X)	P_P=?	0	0

d) What will be the no-arbitrage price at t = 0 of a European put option, if that put option has as underlying asset, asset X, a maturity of t = 1 and an exercise price of 80€? Compute the risk-free interest rate in this economy and check whether put-call parity holds. (Use as actualization factor this one: $(1 + r)^{-T}$).

- Let's first note the payoffs from the put in each state. They are zero!
- You would never exercise this put because you can always sell at a higher price
- So the price of the put is also zero, $P_P = 0$ (if you like, think of this as the cost of the replicating portfolio with zero positions in X and Y)

1. Part d) solution

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
Call, K=80 (C_X)	21.778	70	10
Put, K=80 (P_X)	0	0	0

d) What will be the no-arbitrage price at t = 0 of a European put option, if that put option has as underlying asset, asset X, a maturity of t = 1 and an exercise price of 80€? Compute the risk-free interest rate in this economy and check whether put-call parity holds. (Use as actualization factor this one: $(1+r)^{-T}$).

- We know that $P_B = 146.67$. So the risk-free rate is

$$r = \frac{150 - 146.67}{146.67} = 0.0227 = 2.27\%$$

- Put-call parity:

$$\underbrace{21.778}_{c} + \underbrace{\frac{80}{1.0227}}_{K/(1+r)^{T-t}} = 100 = \underbrace{0}_{p} + \underbrace{100}_{S_t}$$

1. Part e)

	t=0	t=1; state A	t=1; state B
Asset X	100	150	90
Asset Y	10	50	0
Long Future F_X	0	150-F	90-F

e) Compute the price of a futures contract on one unit of asset X with maturity of 1 year.

- Spot-forward parity:

$$F_X = S_{X,0}(1 + r) = 100 * 1.0227 = 102.27$$

2. Exam September 2012.

Consider a one-period economy (dates $t = 0$ and $t = 1$) in which initially there are two assets (asset X and asset Y). At $t = 1$ the economy can be in either two states of nature (state A and state B). The probability of each state is 0.5. The evolution of the prices of these two assets over time conditional on the states of nature is the following:

	t=0	t=1; state A	t=1; state B
Asset X	100	130	90
Asset Y	10	30	0

There is an organized market for both assets where they can be sold and purchased (short-selling is also allowed) at any point in time at the prices quoted above.

a) If a riskless zero-coupon bond is created with a face value of €100 and maturity at $t = 1$, what would be its no-arbitrage price?

2. a).

	t=0	t=1; state A	t=1; state B
Asset X	100	130	90
Asset Y	10	30	0
Bond	P_B =?	100	100

- Replicating portfolio for the bond:

$$\text{State A:} \quad 130zX + 30zY = 100 \tag{5}$$
$$\text{State B:} \quad 90zX = 100 \tag{6}$$

$$\text{From (6)}: \quad zX = 1.1111$$
$$\text{Plugging into (5)}: \quad zY = \frac{100 - 130 * 1.1111}{30} = -1.4815$$

Replicating portfolio Z: long 1.1111 units of X and short 1.4815 units of Y

- Bond price equals the price of this replicating portfolio

$$P_B = 100 * 1.1111 + 10 * (-1.4815) = 96.30$$

2. b).

	t=0	t=1; state A	t=1; state B
Asset X	100	130	90
Asset Y	10	30	0
Call C_X	8	30	0

b) Suppose that a European CALL option on asset X is created; its time to expiration is one year from now and its exercise price is €100. The call is currently trading in the market at €8. Discuss whether there is an arbitrage opportunity or not. If so, describe an arbitrage strategy detailing the assets and also the amounts you would buy and sell.

- To determine the no-arbitrage price, let's compute the state-contingent payoffs for C_X and construct a replicating portfolio
- Payoffs are $max[130 - 100, 0] = 30$ in State A and $max[90 - 100, 0] = 0$ in State B
- The replicating portfolio is asset Y! So unless $P_C = P_Y$, there is arbitrage

2. b).

	t=0	t=1; state A	t=1; state B
Short Y	10	-30	0
Long Call C_X	-8	30	0
Payoff	2	0	0

- $P_C < P_Y$. So the call is too cheap.
- Arbitrage strategy: long call, short Y. Make a profit of 2 today at zero cost in the future

2. c).

	t=0	t=1; state A	t=1; state B
Asset X	100	130	90
Asset Y	10	30	0
Call C_X K=100	10	30	0
Put P_X K=100	$P_P = ?$	0	10

c) Suppose that a European CALL option on asset X is created; its time to expiration is one year from now and its exercise price is €80. Obtain its payoffs at t = 1 and its current theoretical price.

- Payoffs are $max[130 - 80, 0] = 50$ in State A and $max[90 - 80, 0] = 10$ in State B
- Solving for the replicating portfolio:

$$\text{State A:} \quad 130zX + 30zY = 50 \tag{7}$$

$$\text{State B:} \quad 90zX = 10 \tag{8}$$

$$\text{From (8)}: \quad zX = 0.1111$$

$$\text{Plugging into (7)}: \quad zY = \frac{50 - 130 * 0.1111}{30} = 1.1852$$

Theoretical price: $0.1111 * 100 + 1.1852 * 10 = 22.96$

2. d).

	t=0	t=1; state A	t=1; state B
Asset X	100	130	90
Asset Y	10	30	0
Call C_X K=100	10	30	0
Put P_X K=100	$P_P = ?$	0	10

d) What is the price of no arbitrage in t = 0 of an European PUT on asset X with one year to expiration and exercise price of €100?

- From part a), we can calculate r and use the put-call parity:

$$r = (100 - 96.3)/96.3 = 0.0384$$

$$\underbrace{10}_{C_X} + \underbrace{\frac{100}{1.0384}}_{K/(1+r)^{T-t}} = P_X + \underbrace{100}_{S_0}$$

$$P_X = 6.3$$

- Note: another way to solve this is to construct a replicating portfolio of $zX = 0.11$ and $zY = -0.48$, and work out the price of this portfolio

3. Exam July 2018.

Consider a one-period economy with two dates, t=0 and t=1, and two states of nature, s=1 and s=2. At t=0, it is possible to buy asset X whose price is 116 and whose payoff at t=1 can be 120 euros (s=1) or 100 euros (s=2). It is also possible to borrow or lend over one year at a rate of 2%. A call option on X, with maturity in one year and strike price K= 110 euros has a price of 8 euros. Explain if there is an arbitrage opportunity and if so, describe in detail the arbitrage strategy and the cash-flows in each period (t=0 and t=1).

	t=0	t=1; state 1	t=1; state 2
Asset X	116	120	100
Bond	$1/1.02 = 0.98$	1	1
Call C_X K=110	8	10	0

- Payoffs of the call option are: $max[120 - 110, 0] = 0$ in state 1 and $max[100 - 110, 0] = 0$ in state 2

3. Exam July 2018.

	t=0	t=1; state 1	t=1; state 2
Asset X	116	120	100
Bond	$1/1.02 = 0.98$	1	1
Call C_X K=110	8	10	0

- Solving for the replicating portfolio:

$$\text{State A:} \quad 120zX + zB = 10 \tag{9}$$
$$\text{State B:} \quad 100zX + zB = 0 \tag{10}$$

$$\text{(10) from (9)}: \quad 20zX = 10 \quad \Rightarrow \quad zX = 0.5$$
$$\text{Plugging into (10)}: \quad zB = -100zX = -50$$

- No-arbitrage price: $P_C^{NA} = 0.5 * 116 - 50 * 0.98 = 9$
- Actual price: $P_C = 8 < P_C^{NA}$
- The call is too cheap. Arbitrage strategy: long call, short replicating portfolio

3. Exam July 2018.

	t=0	t=1; state 1	t=1; state 2
Short Z: Short 0.5 X, long 50x Bond	9	-10	0
Long Call C_X K=110	-8	10	0
Payoff	1	0	0

- The call is too cheap. Arbitrage strategy: long call, short replicating portfolio Z
- This strategy gives a profit of €1 today at zero cost in the future

4. Exam July 2018.

Suppose a stock that pays no dividend is traded in the market at price $S_0 = 100$€. The one-year risk-free interest is 1%. If a futures contract on the stock is traded at 102€, explain if there is an arbitrage opportunity and if so, describe in detail the arbitrage strategy and the cash-flows in each period (t=0 and t=1).

- First, we work out the no-arbitrage price of the future using the spot-forward parity:

$$\text{No-arbitrage price:} \quad F^{NA} = S_0(1+r) = 100 * 1.01 = 101$$

- Actual price $F = 102 > F^{NA} = 101$. The future is too expensive.
- Arbitrage strategy: short future, borrow at r to buy stock

4. Exam July 2018.

	t=0	t=1
Position 1:		
Buy stock	-100	S_1
Borrow PV of 102	$102/1.01 = 100.99$	-102
Total Position 1	2	$S_1 - 102$
Position 2:		
Short future	0	$102 - S_1$
Total Payoff	0.99	0

- Arbitrage strategy: short future, borrow at r to buy stock
- Profit of €0.99 today at zero cost in the future